ZEILBERGER'S ALGORITHM - EXISTENCE OF THE TELESCOPED RECURRENCE

MAT625 SEMINAR: AUTOMATIC PROOFS OF BINOMIAL SUM IDENTITIES

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Note that in the following, I always assume that $0 \in \mathbb{N}$. All statements, proofs, ideas and remarks are taken from or based on sections 3.2, 4.4, 6.1 and 6.2 of the book $A = B$ by Marko Petkovšek, Herbert S. Wilf and Doron Zeilberger (A K Peters, Ltd., 1996).

1. Setting

We consider a sum of the form

$$f(n) = \sum_k F(n,k),$$

for n, k such that the terms $F(n,k)$ are well-defined and hypergeometric.

Definition. *A term $F(n,k)$ is a* ***hypergeometric term*** *in both arguments, if*

$$\frac{F(n+1,k)}{F(n,k)} \qquad \text{and} \qquad \frac{F(n,k+1)}{F(n,k)}$$

are both rational functions of n and k.

The aim of this considerations is a proof, which provides (under the further little assumption that the terms $F(n,k)$ are even *proper hypergeometric*) the existence of some index $J \in \mathbb{N}_{>0}$, polynomials $\left(a_j\right)_{j=0}^{J}$ in $\mathbb{C}[n]$, not all zero, and a term $G(n,k)$ such that

$$\sum_{j=0}^{J} a_j(n) F(n+j,k) = G(n,k+1) - G(n,k)$$

(called a **telescoped recurrence**).
The motivation behind this proof is based on the fact, that Zeilberger's algorithm relies on this existence result. In more details, it doesn't use it in a constructive way, but the existence is important to be sure that the algorithm is deterministic. In a bigger picture, such a telescoped recurrence is desirable to have, since if we assume G to have finite support, summing both sides over k and dividing by the size of the support of $G(n,k)$ as a function of k leads to

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$$\sum_{j=0}^{J} a_j(n) f(n+j) = 0.$$

Now (as explained in many details in Noam Szyfer's report), there are several cases: $J = 0$, $J = 1$ and $J > 1$. If $J < 2$, it is possible to find directly a closed form for $f(n)$. For the case when $J > 1$, it is also possible to find directly a closed form, if the coefficients $a_j(n)$ are constant (which in this case corresponds to solving a linear recurrence). All other cases can be handled by other algorithms. To summarize: If we can find a telescoped recurrence and G has finite support, we are able to solve the problem to write $f(n)$ in a closed form.

2. Statement

Definition. *A term $F(n,k)$ is **proper hypergeometric** if it can be written in the form*

$$F(n,k) = P(n,k) \frac{\prod_{i=1}^{q} (a_i n + b_i k + c_i)!}{\prod_{j=1}^{r} (u_i n + v_i k + w_i)!} x^k,$$

where $x \in \mathbb{C}$, $P \in \mathbb{C}[n,k]$, $a_i, b_i, u_j, v_j \in \mathbb{Z}$ for all $i \in \{1, ..., r\}, j \in \{1, ..., q\}$ and $r, q \in \mathbb{Z}_{\geq 0}$.

Note that clearly every proper hypergeometric term $F(n,k)$ is hypergeometric. In fact, for the existence of a recurrence in telescoped form, we will assume that $F(n,k)$ is *proper hypergeometric*. The reason for this is, that we want to be able to apply the Fundamental Theorem about the existence of a 2-variable recurrence, and then get our 1-variable recurrence from that.
Precisely, we will prove the following main statement:

Theorem. *Let $F(n,k)$ be a proper hypergeometric term. Then there are $J \in \mathbb{N}_{>0}$, polynomials $(a_j)_{j=0}^{J}$ in $\mathbb{C}[n]$, not all zero, and a function $G(n,k)$ such that (whenever $F(n,k) \neq 0$ and all appearing terms of the form $F(n+j,k)$ are well-defined)*

$$\sum_{j=0}^{J} a_j(n) F(n+j,k) = G(n,k+1) - G(n,k)$$

and $\frac{G(n,k)}{F(n,k)}$ is a rational function.

3. Preparation for the proof

Theorem. *(Fundamental theorem, first part)*
Let $F(n,k)$ be a proper hypergeometric term. Then there exist $I, J \in \mathbb{N}_{>0}$ and polynomials $a_{ij} \in \mathbb{C}[n]$ for $i = 0, ..., I, j = 0, ..., J$, not all zero, such that

$$\sum_{i=0}^{I} \sum_{j=0}^{J} a_{ij}(n) F(n-j,k-i) = 0$$

whenever $F(n,k) \neq 0$ and all appearing terms of the form $F(n-j,k-i)$ are well-defined.

Definition. *(Shift Operators)*
If $p(n)$ (resp. $u(k)$) is a term dependent on n (resp. k), then define

$$N\big(p(n)\big) := p(n+1) \qquad \text{and} \qquad K\big(u(k)\big) := u(k+1)\,.$$

I will use (as in the book) some "distributive notation", i.e.

$$\Big(aN^nK^k + bK^lN^m + c\Big)\big(F(n,k)\big) := aN^n\Big(K^k\big(F(n,k)\big)\Big) + bK^l\Big(N^m\big(F(n,k)\big)\Big) + cF(n,k)$$

for $a,b,c \in \mathbb{C}, n,k,l,m \in \mathbb{N}$.

Lemma 1. *Let $P \in \mathbb{C}[u,v,w]$ be a polynomial. Then there exists a polynomial $Q \in \mathbb{C}[u,v,w]$ such that*

$$P(u,v,w) = P(u,v,1) + (1-w)\,Q(u,v,w)\,.$$

Proof. *Idea:* Consider P as a polynomial of one variable w (u, v are parameters) and consider its Taylor series in $w = 1$
Let $P \in \mathbb{C}[u,v,w]$ be a polynomial. Its Taylor series in $w = 1$ is

$$\sum_{n=0}^{\infty} \frac{\frac{\partial^n P}{\partial w^n}(u,v,1)}{n!}(w-1)^n\,,$$

a finite sum (since P is a polynomial). Thus there exists a polynomial $Q \in \mathbb{C}[u,v,w]$ such that

$$P(u,v,w) = P(u,v,1) + (1-w)\,Q(u,v,w)\,,$$

i.e. choose

$$Q(u,v,w) := -\left(\sum_{n=1}^{\infty} \frac{\frac{\partial^n P}{\partial w^n}(u,v,1)}{n!}(w-1)^{n-1}\right).$$

Alternatively: Consider the polynomial divison of $P(u,v,w)$ by $(1-w)$. There exist polynomials $T, Q \in \mathbb{C}[u,v,w]$ such that

$$P(u,v,w) = T(u,v,w) + (1-w)\,Q(u,v,w)$$

and the degree of T in w is 0 (since $\deg(1-w) = 1$). Thus T is constant in w. By evaluating both sides in $w = 1$, we immediately get

$$P(u,v,1) = T(u,v,1) = T(u,v,w)\,,$$

which shows the lemma. ■

Lemma 2. *Let $Q \in \mathbb{C}[x,y,z]$ and $F(n,k)$ be a hypergeometric term in both arguments. Then $Q(N,n,K)\,F(n,k)$ is a rational multiple of $F(n,k)$.*

Proof. Let $F(n,k)$ be a hypergeometric term in both arguments and $Q \in \mathbb{C}[x,y,z]$. Then $\frac{Q(N,n,K)F(n,k)}{F(n,k)}$ can be written in the form

$$\sum_{(i,j)\in A} a_{ij}(n) \frac{F(n+i,k+j)}{F(n,k)}$$

for some finite $A \subset \mathbb{N}^2$ and $a_{ij} \in \mathbb{C}[n]$ for all $(i,j) \in A$. Note that for each $(i,j) \in A$

$$F(n+i,k+j) = \frac{F(n+i,k+j)}{F(n+i-1,k+j)} F(n+i-1,k+j)$$

and $\frac{F(n+i,k+j)}{F(n+i-1,k+j)}$ is a rational function. Thus (by iterating this procedure) there is some rational function $R(n,k)$ such that

$$F(n+i,k+j) = R(n,k)\, F(n,k+j)\,.$$

With the same argument as before, we find also a rational function $S(n,k)$ such that

$$F(n,k+j) = S(n,k)\, F(n,k)\,.$$

Thus

$$\frac{F(n+i,k+j)}{F(n,k)} = R(n,k)\, S(n,k)\,.$$

Since rational functions multiplied by a polynomial and sums of rational functions are rational functions, the claim follows. ■

4. Proof

The proof is constructive, i. e. it gives an algorithm for finding a recurrence in telescoped form. However, it is not used in Zeilberger's algorithm, since there is a much faster alternative. We only need the proven existence of such a recurrence (to apply Zeilberger's algorithm).
The strategy is the following:

(i) Take the provided 2-variable recurrence from the fundamental theorem.
(ii) Reorder the terms such that we get a recurrence in the desired form which is independent of k.
(iii) Show (from this form) that the provided function $G(n,k)$ is a rational multiple of $F(n,k)$.
(iv) Prove by contradiction that the found recurrence is nontrivial.

Proof. (i) Let $F(n,k)$ be a proper hypergeometric term. Then, using the first part of the fundamental theorem, there exist $I, J \in \mathbb{N}_{>0}$ and polynomials $a_{ij} \in \mathbb{C}[n]$ for $i \in \{0,...,I\}, j \in \{0,...,J\}$, not all zero, such that

$$\sum_{i=0}^{I}\sum_{j=0}^{J} a_{ij}(n)\, F(n+j,k+i) = 0 \tag{1}$$

whenever $F(n,k) \neq 0$ and all of the values $F(n+j,k+i)$ are well-defined. Using the notion of shift operators, (1) can be written in the form

$$P(N,n,K)\big(F(n,k)\big) = 0 \tag{2}$$

for some polynomial $P \in \mathbb{C}[u,v,w|$.

(ii) The goal in this step is to get rid of K on the left side. Using Lemma 1, it's possible to find a polynomial $Q \in \mathbb{C}[u,v,w]$ such that

$$P(u,v,w) = P(u,v,1) + (1-w)\,Q(u,v,w)\,. \tag{3}$$

Plugging this into (2) yields

$$P(N,n,1)\big(F(n,k)\big) + (1-K)\Big(Q(N,n,K)\big(F(n,k)\big)\Big) = 0$$

which is equivalent to

$$P(N,n,1)\big(F(n,k)\big) = (K-1)\Big(Q(N,n,K)\big(F(n,k)\big)\Big)\,. \tag{4}$$

Observe that the left hand side doesn't vary in k (i.e. it is independent of the shift operator K). Define

$$G(n,k) := Q(N,n,K)\big(F(n,k)\big)\,,$$

then, using (4),

$$P(N,n,1)\big(F(n,k)\big) = (K-1)\,G(n,k) = G(n,k+1) - G(n,k)\,. \tag{5}$$

(iii) Using Lemma 2 (applied to $G(n,k)$), it follows that

$$\frac{G(n,k)}{F(n,k)}$$

is rational.

(iv) It remains to prove, that the recurrence on the left hand side of (5) is non-trivial. For that sake, note first that (from the fundamental theorem) $P(N,n,K) \not\equiv 0$. Choose $P(N,n,K)$ such that it fulfills this property and (2) with the least possible degree in K. Then, using (3), write

$$P(N,n,K) = P(N,n,1) + (1-K)\,Q(N,n,K)\,.$$

Now (for the proof of contradiction) assume that $P(N,n,1) \equiv 0$. Note that (by definition of P), $P(N,n,K)\,F(n,k) = 0$. Therefore $P(N,n,1) \equiv 0$ implies that

$$G(n,k+1) - G(n,k) = (K-1)\Big(Q(N,n,K)\big(F(n,k)\big)\Big) = 0,$$

which shows that (under this assumption) $G(n,k)$ is independent of k. Thus G is a hypergeometric term only dependent on n, i.e.

$$\frac{G(n+1,k)}{G(n,k)} = g(n)$$

where $g(n)$ is a rational function only dependent on n. Thus

$$\big(N - g(n)\big)\big(G(n,k)\big) = 0.$$

Since $g(n)$ is rational there exist $a, b \in \mathbb{C}[n]$ such that

$$g(n) = \frac{a(n)}{b(n)}.$$

Hence

$$\big(b(n)\,N - a(n)\big)\,G(n,k) = 0,$$

i.e. we have found a recurrence of order 1 for $G(n,k)$. Now we distinguish between two cases. First, if $\mathbf{Q} \equiv \mathbf{0}$, then we get immediately a contradiction, since this implies that

$$P(N,n,K) = P(N,n,1)\,,$$

but $P(N,n,K) \not\equiv 0$ and $P(N,n,1) \equiv 0$. For the case where $\mathbf{Q} \not\equiv \mathbf{0}$, note that

$$\big(b(n)\,N - a(n)\big)\,\big(G(n,k)\big) = 0.$$

So,

$$\big(b(n)\,N - a(n)\big)\,\Big(Q(N,n,K)\,\big(F(n,k)\big)\Big) = 0$$

is a nontrivial recurrence for $F(n,k)$, which also yields a contradiction since the degree in K of $\big(b(n)\,N - a(n)\big)\,Q(N,n,K)$ is (by definition of Q) strictly smaller than the degree in K of $P(N,n,K)$.

The conclusion of (i) - (iv) is, that the claimed statement is true and $P(N,n,1)$ is the desired nontrivial recurrence in telescoped form, which always exists (under certain quite general assumptions for $F(n,k)$). ■